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Complete Solutions and Extremality Criteria to Polynomial Optimization Problems

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Abstract. This paper presents a set of complete solutions to a class of polynomial optimization problems. By using the so-called *sequential canonical dual transformation* developed in the author's recent book [Gao, D.Y. (2000), Duality Principles in Nonconvex Systems: Theory, Method and Applications, Kluwer Academic Publishers, Dordrecht/Boston/London, xviii + 454 pp], the nonconvex polynomials in $\mathbb{R}^n$ can be converted into an one-dimensional canonical dual optimization problem, which can be solved completely. Therefore, a set of complete solutions to the original problem is obtained. Both global minimizer and local extrema of certain special polynomials can be indentified by Gao-Strang's gap function and triality theory. For general nonconvex polynomial minimization problems, a sufficient condition is proposed to identify global minimizer. Applications are illustrated by several examples.

Key words: critical point theory, duality, global optimization, nonlinear programming, NP-hard problem, polynomial minimization.

1. Problem and Motivation

We consider polynomial minimization problems of the type (in short, the primal problem $(\mathcal{P})$):

$$\min\{P(\mathbf{x}) = W(\mathbf{x}) - \mathbf{x}^T\mathbf{f} : \mathbf{x} \in \mathbb{R}^n\}, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ is a real vector, $\mathbf{f} \in \mathbb{R}^n$ is a given vector, and $W(\mathbf{x})$ is a polynomial of degree d. It is known that the polynomial minimization problem is NP-hard even when $d = 4$ (see [14]). Due to nonconvexity of the cost function $P(\mathbf{x})$, the problem (1) may possess many local minimizers and it represents a global optimization problem. It is known that the application of traditional local optimization procedures for solving nonconvex problems can not guarantee the identification of the global minima (see [13]). Therefore, many numerical methods and algorithms have been suggested recently for finding the lower bounds of polynomial optimization problems (see [1, 15, 16]).

The primary goal of this paper is to present a potentially useful canonical dual transformation method for solving a special polynomial minimization problem ($\mathcal{P}$) where W is a so-called *canonical polynomial* of degree $d = 2^{p+1}$ (see [5]), defined by

$$W(\mathbf{x}) = \frac{1}{2}\alpha_p\Big(\frac{1}{2}\alpha_{p-1}\Big(\dots\Big(\frac{1}{2}\alpha_1\Big(\frac{1}{2}|\mathbf{x}|^2 - \lambda_1\Big)^2 \dots\Big)^2 - \lambda_{p-1}\Big)^2 - \lambda_p\Big)^2, \tag{2}$$

There α_i, λ_i are given parameters. The nonconvex function W appears in many applications. In the simplest case where $p = 1$,

$$W(\mathbf{x}) = \frac{1}{2}\alpha_1\left(\frac{1}{2}|\mathbf{x}|^2 - \lambda_1\right)^2$$

is the so-called *double-well potential* of the scalar-valued function $u = |\mathbf{x}|$, which was first studied by van der Waals in fluids mechanics in 1895. Particularly, if $n = 2$, $p = 1$, then

$$W(x_1, x_2) = \frac{1}{2}\alpha_1\left(\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \lambda_1\right)^2$$

is the so-called “Mexican hat” function in cosmology and theoretical physics. In solid mechanics where the scalar function $u(\mathbf{x})$ is a field function, then $W(\mathbf{x}) = \frac{1}{2}\alpha_1(\frac{1}{2}u(\mathbf{x})^2 - \lambda_1)^2$ is the well-known second-order Landau potential in phase transitions of superconductivity and shape memory alloys. In post-buckling analysis of extended beam theory developed by the author [7], each potential well of W represents a possible buckled beam state. Numerical discretizations of these mechanics problems usually lead to a large-scale polynomial optimization problems of type ($\mathcal{P}$). The criticality condition $\nabla P(\mathbf{x}) = 0$ gives a coupled, nonlinear algebraic system with n unknown $\mathbf{x} \in \mathbb{R}^n$:

$$\prod_{k=1}^{p} \alpha_k(\xi_k(\mathbf{x}) - \lambda_k)\mathbf{x} = \mathbf{f}, \tag{3}$$

where

$$\xi_0(\mathbf{x}) = |\mathbf{x}|, \quad \xi_k(\mathbf{x}) = \frac{1}{2}\alpha_{k-1}(\xi_{k-1}(\mathbf{x}) - \lambda_{k-1})^2, \quad k = 1, \dots, p, \tag{4}$$

and $\alpha_0 = 1$, $\lambda_0 = 0$. Clearly, direct methods for solving this coupled, nonlinear algebraic system are very difficult. Also Equation (3) is the only a necessary condition for local minima. In this paper, we will present a complete set of solutions to Equation (3) with sufficient conditions for global and local minima by using the *sequential canonical dual transformation*. This method has been successfully applied for solving a large class of nonconvex variational analysis and global optimization problems (see [3,6,10,11]).

2. Complete Solutions

By the use of the sequential canonical dual transformation developed in [5], the perfect dual problem (with zero duality gap) (($\mathcal{P}^d$) in short) of the polynomial optimization ($\mathcal{P}$) can be formulated as the following

$$(\mathcal{P}^d):\ \max_{\varsigma} \left\{ P^d(\varsigma) = -\frac{|\mathbf{f}|^2}{2\varsigma_p!} - \sum_{k=1}^{p} \frac{\varsigma_p!}{\varsigma_k!} W_k^*(\varsigma_k) \right\}, \tag{5}$$

where $\varsigma_p! = \varsigma_p \varsigma_{p-1} \cdots \varsigma_2 \varsigma_1$, and

$$\varsigma_1 = \varsigma, \qquad \varsigma_k = \alpha_k \left(\frac{1}{2\alpha_{k-1}} \varsigma_{k-1}^2 - \lambda_k \right), \quad k = 2, \dots, p. \tag{6}$$

$W_k^*(\varsigma_k)$ is a quadratic function of ς_k defined by

$$W_k^*(\varsigma_k) = \frac{1}{2\alpha_k} \varsigma_k^2 + \lambda_k \varsigma_k.$$

The dual problem is a nonlinear programming with only one variable $\varsigma \in \mathbb{R}$, which is much easier than the primal problem. Clearly, for any $\varsigma \neq 0$ and $\varsigma_k^2 \neq 2\alpha_k \lambda_{k+1}$, the dual function P^d is well defined and the criticality condition $\delta P^d(\varsigma) = 0$ leads to a dual algebraic equation

$$2(\varsigma_p!)^2 (\alpha_1^{-1} \varsigma + \lambda_1) = |\mathbf{f}|^2. \tag{7}$$

THEOREM 1 (Complete Solution Set). *For any given parameters α_k, λ_k ($k = 1, \dots, p$) and the input* $\mathbf{f}$, *the dual algebraic Equation* (7) *has at most $s = 2^{p+1} - 1$ real solutions: $\bar{\varsigma}^{(i)}$ ($i = 1, \dots, s$). For each dual solution $\bar{\varsigma} \in \mathbb{R}$, the vector $\bar{\mathbf{x}}$ defined by*

$$\bar{\mathbf{x}}(\bar{\varsigma}) = (\bar{\varsigma}_p!)^{-1} \mathbf{f} \tag{8}$$

is a critical point of the primal problem ($\mathcal{P}$) *and*

$$P(\bar{\mathbf{x}}) = P^d(\bar{\varsigma}).$$

Conversely, every critical point $\bar{\mathbf{x}}$ of the polynomial $P(\mathbf{x})$ can be written in form (8) *for some dual solution $\bar{\varsigma} \in \mathbb{R}$.*

Proof. We first prove the vector defined by (8) solves (3). Substituting $(\bar{\varsigma}_p!)^{-1}\mathbf{f} = \bar{\mathbf{x}}$ into the dual algebraic Equation (7), we obtain

$$\alpha_1\left(\tfrac{1}{2}|\bar{\mathbf{x}}|^2 - \lambda_1\right) = \alpha_1(\bar{\xi}_1 - \lambda_1) = \bar{\varsigma}. \tag{9}$$

Thus from (6) we have

$$\bar{\varsigma}_k = \alpha_k(\bar{\xi}_k - \lambda_k), \quad k = 1, \ldots, p. \tag{10}$$

Substituting

$$\bar{\mathbf{x}}(\bar{\varsigma}) = (\bar{\varsigma}_p!)^{-1}\mathbf{f} = \left(\prod_{k=1}^{p} \alpha_k(\bar{\xi}_k - \lambda_k)\right)^{-1} \mathbf{f}$$

into the left hand side of the canonical Equation (3) leads to $\mathbf{f}$. Thus for every solution $\bar{\varsigma}$ of the dual algebraic Equation (7), $\bar{\mathbf{x}} = (\bar{\varsigma}_p!)^{-1}\mathbf{f}$ solves the canonical Equation (3), and is a critical point of P.

Conversely, if $\bar{\mathbf{x}}$ is a solution of the couple nonlinear system (3), then it can be written in the form $\bar{\mathbf{x}} = (\bar{\varsigma}_p!)^{-1}\mathbf{f}$ with $\bar{\varsigma}_k = \alpha_k(\bar{\xi}_k - \lambda_k)$, $k = 1, \ldots, p$ and $\bar{\xi}_1 = \frac{1}{2}|\bar{\mathbf{x}}|^2$. Thus in terms of $\bar{\varsigma}_k$, we have

$$\bar{\xi}_1 = \frac{1}{2}|\bar{\mathbf{x}}|^2 = \frac{1}{2}(\bar{\varsigma}_p!)^{-2}|\mathbf{f}|^2 = \frac{1}{\alpha_1}\bar{\varsigma}_1 + \lambda_1.$$

This is the dual algebraic Equation (7), in which $\bar{\varsigma}_k = \alpha_k(\bar{\xi}_k - \lambda_k)$. Since

$$\bar{\xi}_{k+1} = \frac{1}{2}\alpha_k(\bar{\xi}_k - \lambda_k)^2 = \frac{1}{2\alpha_k}\bar{\varsigma}_k^2 = \frac{1}{\alpha_{k+1}}\bar{\varsigma}_{k+1} + \lambda_{k+1},$$

we have

$$\bar{\varsigma}_{k+1} = \alpha_{k+1}\left(\frac{1}{2\alpha_k}\bar{\varsigma}_k^2 - \lambda_{k+1}\right).$$

This shows that every solution of the coupled nonlinear system (3) can be written in the form $\bar{\mathbf{x}} = (\bar{\varsigma}_p!)^{-1}\mathbf{f}$ for some solution $\bar{\varsigma}$ of the dual algebraic Equation (7). □

3. Global and Local Optimality Criteria

This section will provide some sufficient conditions for global and local extrema.

3.1. TRIALITY THEORY FOR CASE $p=1$

The primal problem $(\mathcal{P})$ for $p=1$ is to find all critical points of the nonconvex function

$$P(\mathbf{x}) = \tfrac{1}{2}\alpha_1\left(\tfrac{1}{2}|\mathbf{x}|^2 - \lambda_1\right)^2 - \mathbf{x}^T\mathbf{f}.$$

The canonical dual function for this simple case is

$$P^d(\varsigma) = -\frac{|\mathbf{f}|^2}{2\varsigma} - \frac{1}{2}\alpha_1^{-1}\varsigma^2 - \varsigma\lambda_1.$$

The dual algebraic equation

$$2\varsigma^2(\alpha_1^{-1}\varsigma + \lambda_1) = |\mathbf{f}|^2 \tag{11}$$

has at most three real roots $\bar{\varsigma}^{(i)}$ $(i=1,2,3)$, and the vector $\bar{\mathbf{x}}_i = \mathbf{f}/\bar{\varsigma}^{(i)}$ is a critical point of the nonconvex function $P(\mathbf{x})$. Let $\phi_1(\varsigma) = \pm\varsigma\sqrt{2(\alpha_1^{-1}\varsigma + \lambda_1)}$. In algebraic geometry, the graph of $\phi_1(\varsigma)$ is the so-called singular algebraic curve in $(\varsigma, |\mathbf{f}|)$-space (see Figure 2).

The following theorem reveals the extremality of these critical points.

THEOREM 2 (Triality theorem [5]). *Let $\lambda_1, \alpha_1 > 0$ be two given parameters. If $|\mathbf{f}| < h = \sqrt{8\alpha_1^2\lambda_1^3/27}$, the dual algebraic Equation* (11) *has three real roots satisfying $\bar{\varsigma}^{(1)} > 0 > \bar{\varsigma}^{(2)} \geqslant \bar{\varsigma}^{(3)}$, and the vector $\bar{\mathbf{x}}_1 = \mathbf{f}/\bar{\varsigma}^{(1)}$ is a global minimizer, $\bar{\mathbf{x}}_2 = \mathbf{f}/\bar{\varsigma}^{(2)}$ is a local minimizer, while $\bar{\mathbf{x}}_3 = \mathbf{f}/\bar{\varsigma}^{(3)}$ is a local maximizer. If $|\mathbf{f}| < h$, the dual algebraic Equation* (11) *has a unique root $\bar{\varsigma}^{(1)} > 0$, and the vector $\bar{\mathbf{x}}_1$ is a global minimizer of the function $P(\mathbf{x})$. However, if $|\mathbf{f}| = h$, the dual algebraic Equation* (11) *has only two roots $\bar{\varsigma}^{(1)} > 0 > \bar{\varsigma}^{(2)}$, the vector $\bar{\mathbf{x}}_1 = \mathbf{f}/\bar{\varsigma}^{(1)}$ is a global minimizer of the function $P(\mathbf{x})$, while the vector $\bar{\mathbf{x}}_2 = \mathbf{f}/\bar{\varsigma}^{(2)}$ is a local stationary point.*

REMARK. For $p=1$, the nonconvex function $W(\mathbf{x})$ is a double-well function of $|\mathbf{x}|$. By using the method introduced by Gao and Strang [12], we let $\xi_1 = \Lambda_1(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2$, then $W(\mathbf{x})$ can be written as $W(\mathbf{x}) = W_1(\Lambda_1(\mathbf{x}))$, where $W_1(\xi_1) = \frac{1}{2}\alpha_1(\xi_1 - \lambda_1)^2$ is the canonical function of ξ_1 (see [5]). Its conjugate function can be easily obtained by the Legendre transformation

$$W_1^*(\varsigma) = \{\xi_1\varsigma - W_1(\xi_1) \mid \varsigma = \partial W_1(\xi_1)/\partial\xi_1 = \alpha_1(\xi_1 - \lambda_1)\} = \tfrac{1}{2}\alpha_1^{-1}\varsigma^2 + \lambda_1\varsigma.$$

Thus, replacing $W(\mathbf{x})$ by $W_1(\Lambda_1(\mathbf{x})) = \Lambda_1(\mathbf{x})\varsigma - W_1^*(\varsigma)$, the nonconvex function $P(\mathbf{x})$ can be written in the following so-called extended Lagrange form:

$$L(\mathbf{x}, \varsigma) = \frac{1}{2}|\mathbf{x}|^2\varsigma - \frac{1}{2}\alpha_1^{-1}\varsigma^2 - \varsigma\lambda_1 - \mathbf{x}^T\mathbf{f} \tag{12}$$

which is actually the *generalized complementary energy* studied by Gao and Strang in nonconvex/nonsmooth variational problem [12], and the term $G(\mathbf{x}, \varsigma) = \frac{1}{2}|\mathbf{x}|^2\varsigma$ is the *complementary gap function*. Gao and Strang proved that if $G(\mathbf{x}, \varsigma) \geqslant 0$, i.e. $\varsigma \geqslant 0$ in this finite dimensional case, $L(\mathbf{x}, \varsigma)$ is a saddle function and

$$\min_{\mathbf{x}\in R^n} \max_{\varsigma \geqslant 0} L(\mathbf{x}, \varsigma) = \max_{\varsigma \geqslant 0} \min_{\mathbf{x}\in R^n} L(\mathbf{x}, \varsigma).$$

It is easy to check that $P(\mathbf{x}) = \max_{\varsigma\geqslant 0} L(\mathbf{x}, \varsigma)$, and $P^d(\varsigma) = \min_{\mathbf{x}\in R^n} L(\mathbf{x}, \varsigma)$ if $\varsigma \neq 0$. Thus the condition $G(\mathbf{x}, \varsigma) \geqslant 0$, $\forall \mathbf{x} \in \mathbb{R}^n$ serves as a sufficient condition for global minimizer, and

$$\min_{\mathbf{x}\in R^n} P(\mathbf{x}) = \min_{\mathbf{x}\in R^n} \max_{\varsigma > 0} L(\mathbf{x}, \varsigma) = \max_{\varsigma > 0} P^d(\varsigma). \tag{13}$$

Furthermore, in the study of post-buckling analysis of large deformed beam theory (see [2]), the author discovered that if $G(\mathbf{x}, \varsigma) \leqslant 0$, then $L(\mathbf{x}, \varsigma)$ is a so-called super-Lagrangian. If $(\bar{\mathbf{x}}, \bar{\varsigma})$ is a critical point of $L(\mathbf{x}, \varsigma)$, and $\bar{\varsigma} < 0$, then in the neighborhood of $(\bar{\mathbf{x}}, \bar{\varsigma})$, we have either

$$P(\bar{\mathbf{x}}) = \min_{\mathbf{x}\in R^n} \max_{\varsigma < 0} L(\mathbf{x}, \varsigma) = \min_{\varsigma < 0} \max_{\mathbf{x}\in R^n} L(\mathbf{x}, \varsigma) = P^d(\bar{\varsigma}), \tag{14}$$

or

$$P(\bar{\mathbf{x}}) = \max_{\mathbf{x}\in R^n} \max_{\varsigma < 0} L(\mathbf{x}, \varsigma) = \max_{\varsigma < 0} \max_{\mathbf{x}\in R^n} L(\mathbf{x}, \varsigma) = P^d(\bar{\varsigma}). \tag{15}$$

This set of three relations (13–15) forms a so-called tri-duality theory in nonconvex analysis [4,5], which was discovered first in post-buckling analysis of a large deformed beam theory [2]. The graphs of $P(x)$ and $P^d(\varsigma)$ for $n = 1$ are illustrated by Figure 1, where we can see that $P^d(\varsigma)$ is strictly concave for $\varsigma > 0$, while for $\varsigma < 0$, $P^d(\varsigma)$ is nonconvex with one local minimizer and a local maximizer.

3.2. GLOBAL MINIMIZER FOR GENERAL CASE

For general case $p \geqslant 1$, the global minimizer of the problem $(\mathcal{P})$ can be identified by the following theorem.

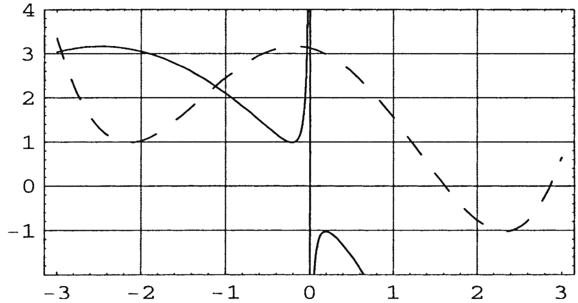


Figure 1. Triality theory: graphs of $P(x)$ (dashed) and $P^d(\varsigma)$ (solid) for $n=1$.

THEOREM 3. *Suppose that for the given positive parameters* $\alpha_k, \lambda_k \geqslant 0\ \forall k \in \{1, \ldots, p\}$, $\bar{\varsigma}$ *is a solution of the dual algebraic Equation* (7). *If*

$$\bar{\varsigma} > \varsigma_+ = \sqrt{2\alpha_1\left(\lambda_2 + \sqrt{\frac{2}{\alpha_2}\left(\lambda_3 + \cdots + \sqrt{\frac{2}{\alpha_{p-2}}\left(\lambda_{p-1} + \sqrt{\frac{2}{\alpha_{p-1}}\lambda_p}\right)}\right)}\right)},$$

then $\bar{\varsigma}$ *is a global maximizer on the open domain* $(\varsigma_+, +\infty)$, $\bar{\mathbf{x}} = (\bar{\varsigma}_p!)^{-1}\mathbf{f}$ *is a global minimizer of* P, *and*

$$P(\bar{\mathbf{x}}) = \min_{\mathbf{x}\in R^n} P(\mathbf{x}) = \max_{\varsigma>\varsigma_+} P^d(\varsigma) = P^d(\bar{\varsigma}). \tag{16}$$

Proof. By using the sequential canonical dual transformation (see [5]), the complementary function associated with the problem $(\mathcal{P})$ is

$$L(\mathbf{x}, \boldsymbol{\varsigma}) = \frac{1}{2}|\mathbf{x}|^2\varsigma_p! - \sum_{k=1}^{p}\frac{\varsigma_p!}{\varsigma_k!}W_k^*(\varsigma_k) - \mathbf{x}^T\mathbf{f}, \tag{17}$$

where $\boldsymbol{\varsigma} = \{\varsigma_1, \ldots, \varsigma_p\} \in \mathbb{R}^p$. It is easy to see that if $\boldsymbol{\varsigma} > 0$, i.e. $\varsigma_k > 0\ \forall k \in \{1, \ldots, p\}$, the Lagrangian L is convex in $\mathbf{x} \in \mathbb{R}^n$ and concave in each ς_k $(k=1, \ldots, p)$. Thus, by the saddle-point theory (see [5]), we have

$$\min_{\mathbf{x}\in R^n} P(\mathbf{x}) = \min_{\mathbf{x}\in R^n}\max_{\boldsymbol{\varsigma}>0} L(\mathbf{x}, \boldsymbol{\varsigma}) = \max_{\boldsymbol{\varsigma}>0}\min_{\mathbf{x}\in R^n} L(\mathbf{x}, \boldsymbol{\varsigma}) = \max_{\boldsymbol{\varsigma}>0} P_p^d(\boldsymbol{\varsigma}),$$

where

$$P_p^d(\boldsymbol{\varsigma}) = -\frac{|\mathbf{f}|^2}{2\varsigma_p!} - \sum_{k=1}^{p}\frac{\varsigma_p!}{\varsigma_k!}W_k^*(\varsigma_k)$$

is concave for each $\varsigma_k > 0$ $(k = 1, 2, \ldots, p)$. The criticality condition $\delta_{\varsigma_k} P_p^d(\boldsymbol{\varsigma}) = 0$ leads to Equation (6). Thus, under the condition $\varsigma > \varsigma_+$,

$$\min_{\mathbf{x} \in \mathbb{R}^n} P(\mathbf{x}) = \max_{\boldsymbol{\varsigma} > 0} P_p^d(\boldsymbol{\varsigma}) = \max_{\varsigma > \varsigma_+} P^d(\varsigma).$$

This proves (16). □

4. Applications

In this section, we present applications of the general theory obtained in this paper to the following cases.

4.1. CASE $p = 1$

We simply let $\alpha_1 = 3$, $\lambda_1 = 3/2$, which gives $h = 3.0$. If we choose $\mathbf{f} = \{5, -3\}/\sqrt{2}$, then $|\mathbf{f}| < h$ and the dual algebraic Equation (11) has only one real root $\varsigma_1 = 1.93 > 0$. By Theorem 2 we know that $\mathbf{x}_1 = \mathbf{f}/\varsigma_1 = \{1.46421, -1.46421\}$ is a global minimizer and $P(\mathbf{x}_1) = -7.66 = P^d(\varsigma_1)$ (Figure 2).

For $\mathbf{f} = \{3, -3\}/\sqrt{2}$, we have $|\mathbf{f}| = h$ and the dual algebraic Equation (11) has two real roots $\varsigma_1 = 1.5 > 0 > \varsigma_2 = -3 = \varsigma_3$. By Theorem 2 we know that $\mathbf{x}_1 = \mathbf{f}/\varsigma_1 = \{1.41421, -1.41421\}$ is a global minimizer, $\mathbf{x}_2 = \mathbf{f}/\varsigma_2 = \{-0.707107, 0.707107\}$ is a local stationary point. It is easy to verify that

$$P(\mathbf{x}_1) = P^d(\varsigma_1) = -5.63 < P(\mathbf{x}_2) = P^d(\varsigma_2) = 4.5.$$

If we choose $\mathbf{f} = \{1, -2\}/\sqrt{2}$, then $|\mathbf{f}| < h$ and the dual algebraic Equation (11) has three real roots $\varsigma_1 = 0.838147 > 0 > \varsigma_2 = -1.04125 > \varsigma_3 = -4.29689$. By Theorem 2 we know that $\mathbf{x}_1 = \mathbf{f}/\varsigma_1 = \{0.843655, -1.68731\}$ is a global minimizer, $\mathbf{x}_2 = \mathbf{f}/\varsigma_2 = \{-0.679092, 1.35818\}$ is a local minimizer, and $\mathbf{x}_3 = \mathbf{f}/\varsigma_3 = \{-0.164562, 0.329125\}$ is local maximizer. It is easy to verify that

$P(\mathbf{x}_1) = P^d(\varsigma_1) = -2.87 < P(\mathbf{x}_2) = P^d(\varsigma_2) = 2.58 < P(\mathbf{x}_3) = P^d(\varsigma_3) = 3.66$. (see Figure 3).

4.2. CASE $p = 2$

In this case, the dual function has the form

$$P^d(\varsigma) = -\frac{|\mathbf{f}|^2}{2\varsigma\varsigma_2} - \left(\frac{1}{\alpha_2}\varsigma_2^2 + \lambda_2\varsigma_2 + \varsigma_2\left(\frac{1}{2\alpha_1}\varsigma^2 + \lambda_1\varsigma\right)\right). \tag{18}$$

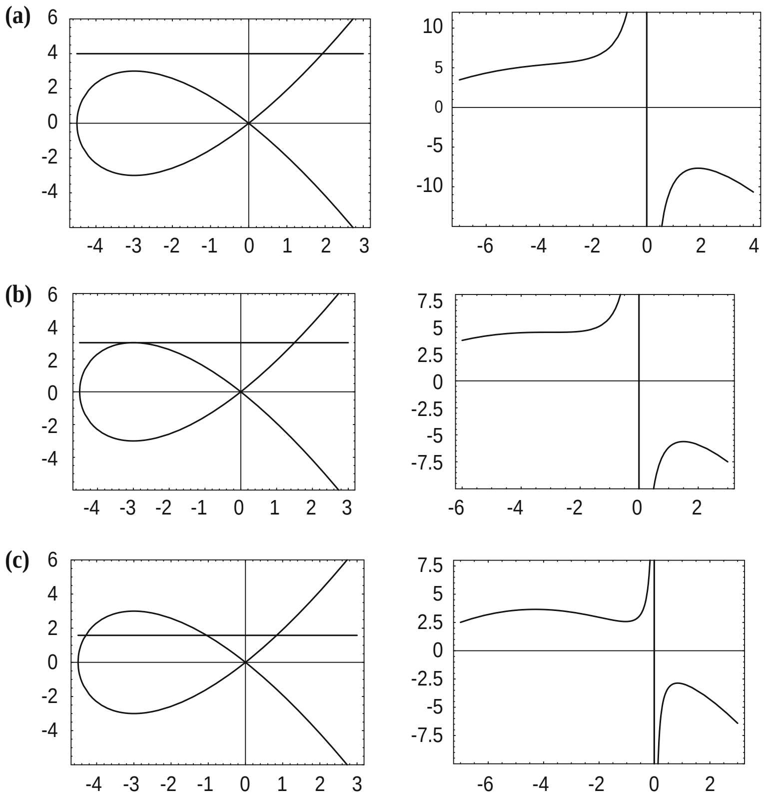


Figure 2. Algebraic curves $|\mathbf{f}| = \phi_1(\varsigma)$ (left) and graphs of dual function P^d (right). (a) $|\mathbf{f}| > h$: Unique solution. (b) $|\mathbf{f}| = h$: two solutions. (c) $|\mathbf{f}| < h$: three solutions.

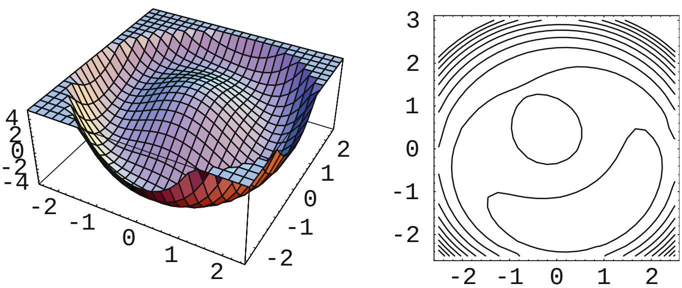


Figure 3. Graph of $P(\mathbf{x})$ with three critical points: global minimizer $\mathbf{x}_1 = \{0.84, -1.69\}$, local minimizer $\mathbf{x}_2 = \{-0.68, 1.36\}$, and local maximizer $\mathbf{x}_3 = \{-0.16, 0.33\}$.

Substituting $\varsigma_2 = \frac{\alpha_2}{2\alpha_1}\varsigma^2 - \lambda_2\alpha_2$ into (7), the dual algebraic equation

$$2\varsigma^2 \left(\frac{\alpha_2}{2\alpha_1}\varsigma^2 - \lambda_2\alpha_2 \right)^2 \left(\frac{1}{\alpha_1}\varsigma + \lambda_1 \right) = |\mathbf{f}|^2 \tag{19}$$

has at most seven real roots $\bar{\varsigma}_i$ $(i=1,\ldots,7)$. Let

$$\phi_2(\varsigma)=\pm\varsigma\left(\frac{\alpha_2}{2\alpha_1}\varsigma^2-\lambda_2\alpha_2\right)\sqrt{2\left(\frac{1}{\alpha_1}\varsigma+\lambda_1\right)},$$

and $\mathbf{f}=\{0.5,-0.2\}$, $\alpha_1=2$, $\alpha_2=1$, and $\lambda_2=1$. Then for different values of λ_1 the graphs of $\phi_2(\varsigma)$ and $P^d(\varsigma)$ are shown in Figure 4. The graphs of $P(\mathbf{x})$ are shown in Figure 5 (for $\lambda_1=0$ and $\lambda_1=1$) and Figure 6 (for $\lambda_1=2$). Since $\varsigma_+=\sqrt{2\alpha_1\lambda_2}=2$, we can see that the dual function $P^d(\varsigma)$ is strictly concave for $\varsigma>\varsigma_+=2$. The dual algebraic Equation (19) has a total of seven real solutions when $\lambda_1=2$, and the biggest $\varsigma_1=2.10>\varsigma_+=2$ gives the global minimizer $\mathbf{x}_1=\mathbf{f}/\varsigma_1=\{2.29,-0.92\}$, and $P(\mathbf{x}_1)=-1.32=P^d(\varsigma_1)$. The smallest $\varsigma_7=-4.0$ gives a local maximizer $\mathbf{x}_7=\{-0.04,0.02\}$ and $P(\mathbf{x}_7)=4.51=P^d(\varsigma_7)$ (see Figure 6).

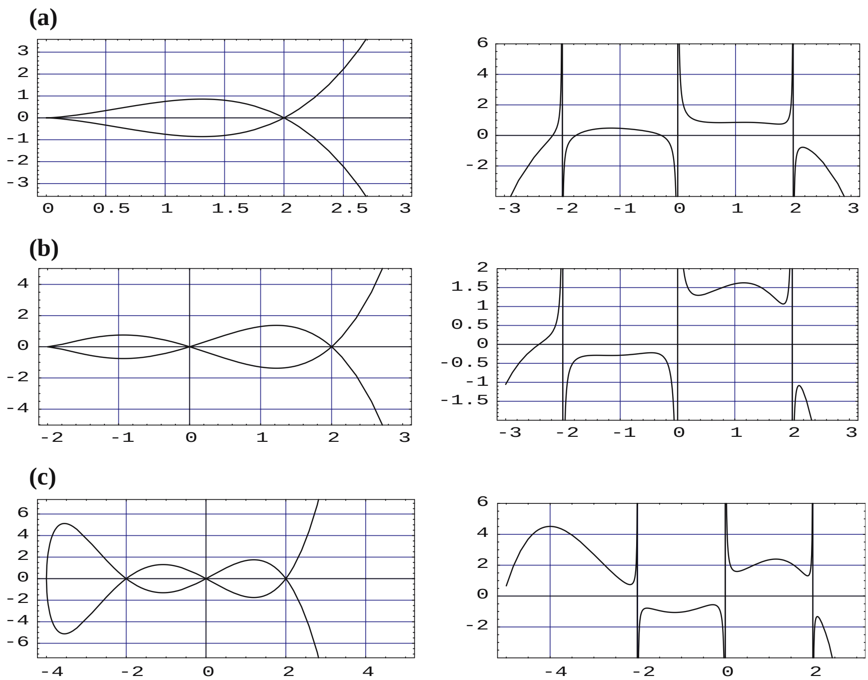
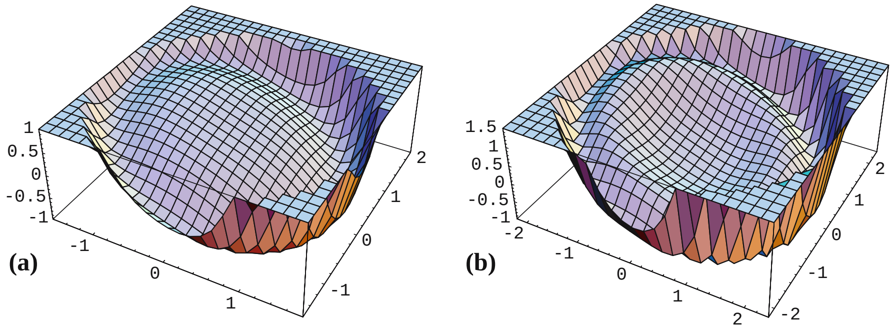


Figure 4. Graphes of the algebraic curve $\phi_2(\varsigma)$ (left) and dual function $P^d(\varsigma)$ (right) (a) $\lambda_1=0$: Three solutions $\varsigma_3=0.73<\varsigma_2=1.75<\varsigma_1=2.16$. (b) $\lambda_1=1$: Five solutions $\{-1.42,-0.46,0.36,1.85,2.12\}$. (c) $\lambda_1=2$: Seven solutions $\{-4.0,-2.18,-1.79,-0.29,0.27,1.88,2.10\}$.



$\psi_1 = 0.$ $\psi_1 = 1.$

Figure 5. Graphs of $P(\mathbf{x})$. (a) $\lambda_1 = 0$. (b) $\lambda_1 = 1$.

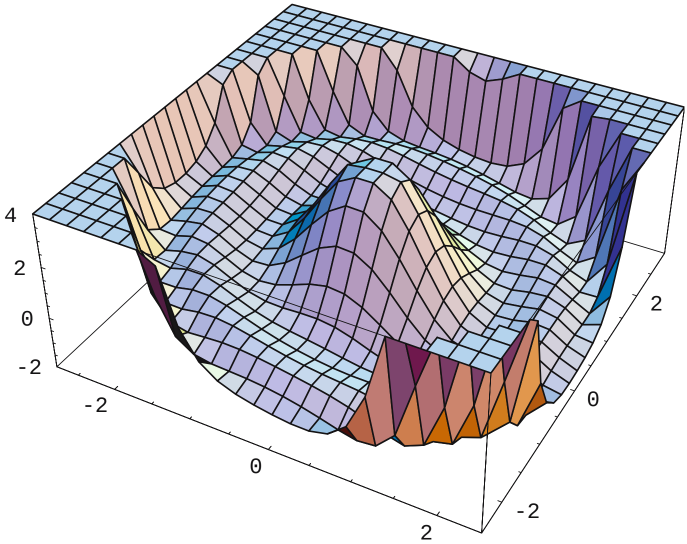


Figure 6. Graph of $P(\mathbf{x})$ with $\lambda_1 = 2$.

4.3. CASE $p = 3$

For $p = 3$, the nonconvex function

$$P(\mathbf{x}) = \frac{1}{2}\alpha_3 \left(\frac{1}{2}\alpha_2 \left(\frac{1}{2}\alpha_1 \left(\frac{1}{2}|\mathbf{x}|^2 - \lambda_1 \right)^2 - \lambda_2 \right)^2 - \lambda_3 \right)^2 - \mathbf{x}^T \mathbf{f}$$

is a polynomial of degree $d = 2^{3+1} = 16$. The dual function has the form

$$P^d(\varsigma) = -\frac{|\mathbf{f}|^2}{2\varsigma\varsigma_2\varsigma_3} - \left(\frac{1}{\alpha_3}\varsigma_3^2 + \lambda_3\varsigma_3 + \varsigma_3 \left(\frac{1}{\alpha_2}\varsigma_2^2 + \lambda_2\varsigma_2 \right) + \varsigma_3\varsigma_2 \left(\frac{1}{2\alpha_1}\varsigma^2 + \lambda_1\varsigma \right) \right), \tag{20}$$

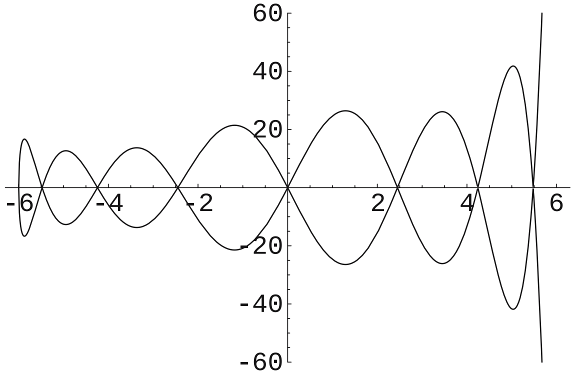


Figure 7. Graph of $\phi_3(\varsigma)$.

where $\varsigma_2 = \frac{\alpha_2}{2\alpha_1}\varsigma^2 - \lambda_2\alpha_2$, $\varsigma_3 = \frac{\alpha_3}{2\alpha_2}\varsigma_2^2 - \lambda_3\alpha_3$. The criticality condition $\delta P^d(\varsigma) = 0$ leads to the dual algebraic equation

$$\phi_3^2(\varsigma) = |\mathbf{f}|^2, \tag{21}$$

where

$$\phi_3(\varsigma) = \pm\varsigma\left(\frac{\alpha_2}{2\alpha_1}\varsigma^2 - \lambda_2\alpha_2\right)\left(\frac{\alpha_3}{2\alpha_2}\left(\frac{\alpha_2}{2\alpha_1}\varsigma^2 - \lambda_2\alpha_2\right)^2 - \lambda_3\alpha_3\right)\sqrt{2\left(\frac{1}{\alpha_1}\varsigma + \lambda_1\right)}.$$

If we choose $\alpha_1 = 3$, $\alpha_2 = 1$, $\alpha_3 = 2$ and $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 2$, the graph of $\phi_3(\varsigma)$ is shown in Figure 7. In this case,

$$\varsigma_+ = \sqrt{2\alpha_1\left(\lambda_2 + \sqrt{\frac{2}{\alpha_2}\lambda_3}\right)} = 5.48.$$

Particularly, if we let $\mathbf{f} = \{1, -1\}$, the dual problem has a unique solution $\varsigma_1 = 5.48355$ on the domain (ς_+, ∞), which leads to a global minimizer $\mathbf{x}_1 = \{1.95649, -1.95649\}$, and we have $P(\mathbf{x}_1) = -3.912 = P^d(\varsigma_1)$.

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